Answer sheet for Assignment 02 – MPZ 3132

(01) (a)

i). In symbolic form the given equation may be written as

$(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$

The auxiliary equation is

$(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$

$\therefore m = 2{,}2$ (05)

So complementary function is

$\boldsymbol{Y_c = (C_1 + C_2 x)e^{2x}}$ (05)

Where C_1, C_2 are arbitrary constants.

Particular Integral is

$$Y_p = \frac{x^2 + e^x + \cos 2x}{(D^2 - 4D + 4)}$$

$$= \frac{1}{(D-2)^2}x^2 + \frac{e^x}{(D-2)^2} + \frac{\cos 2x}{(D-2)^2}$$

$$= \frac{x^2}{\left\{-2\left(1 - \frac{D}{2}\right)\right\}^2} + \frac{e^x}{(1-2)^2} + \frac{\cos 2x}{(D^2 - 4D + 4)}$$

$$Y_p = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2 + e^x - \frac{1}{4D}\cos 2x$$

Putting $(D^2 = -2^2)$

$$Y_p = \frac{1}{4}\left(1 + 2\frac{D}{2} + 3\frac{D^2}{4}\right)x^2 + e^x - \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{4}\left(x^2 + 2x + 2\frac{3}{4}\right) + e^x - \frac{1}{4}\frac{\sin 2x}{2}$$

$$Y_p = \frac{1}{4}x^2 + \frac{x}{2} + \frac{3}{8} + e^x - \frac{1}{8}\sin 2x \ (30)$$

Hence, the required solution of the given equation is

$Y = Y_p + Y_c$

$$\boldsymbol{Y = (C_1 + C_2 x)e^x + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + e^x - \frac{1}{8}\sin 2x} \qquad \boldsymbol{(05)}$$

ii).

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 1 + x + x^2$$

In symbolic form the given equation may be written as

$(D^2 - 6D + 9)y = 1 + x + x^2$

The auxiliary equation is

$(m^2 - 6m + 9) = 0$

$(m-3)^2 = 0$

$\therefore m = 3,3$ (05)

So complementary function is

$\boldsymbol{Y_c = (C_1 + C_2 x)e^{3x}}$ (05)

Where C_1, C_2 are arbitrary constants.

Particular Integral is

$$Y_p = \frac{1 + x + x^2}{(D^2 - 6D + 9)}$$

$$= \frac{1}{(D-3)^2}x^2 + \frac{x}{(D-3)^2} + \frac{1}{(D-3)^2}$$

$$= \frac{1 + x + x^2}{\left\{-3\left(1 - \frac{D}{3}\right)\right\}^2}$$

$$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2} (1 + x + x^2)$$

$$= \frac{1}{9}\left(1 + 2\frac{D}{3} + 3\frac{D^2}{9} + \cdots\right)(1 + x + x^2)$$

$$= \frac{1}{9}\left\{(1 + x + x^2) + \frac{2}{3}(1 + 2x) + \frac{1}{3}(2)\right\}$$

$$= \frac{1}{9}x^2 + \frac{13}{9}x + \frac{13}{9}(20)$$

Hence, the general equation is

$$\boldsymbol{Y = (C_1 + C_2 x)e^{3x} + \frac{x^2}{9} + \frac{13}{9}x + \frac{13}{9}} \qquad \mathbf{(05)}$$

iii).

$$(D^2 - 2D + 1)y = x^2 e^x$$

The auxiliary equation is

$(m^2 - 2m + 1) = 0$

$(m-1)^2 = 0$ (05)

So complementary function is

$\boldsymbol{Y_c = (C_1 + C_2 x)e^{x}}$ (05)

Where C_1, C_2 are arbitrary constants.

Particular Integral is

$$Y_p = \frac{x^2 e^x}{(D^2 - 2D + 1)}$$

$$= e^x \frac{1}{[(D+1)^2 - 2(D+1) + 1]}x^2$$

$$= e^x \frac{1}{D^2}x^2$$

$$= e^x \frac{1}{D}\frac{x^3}{3}$$

$Y_p = \ e^x \left(\frac{x^4}{12}\right)$ (10)

Hence, the general equation is

$\boldsymbol{Y = (C_1 + C_2 x)e^x + \ e^x \left(\frac{x^4}{12}\right)}$ (05)

iv).

$\frac{d^2y}{dx^2} + y = Cos\ x + e^x \sin x + xe^{2x}$

$(D^2 + 1)y = \ Cos\ x + e^x \sin x + xe^{2x}$

The auxiliary equation is

$(m^2 + 1)y = 0$

$\therefore m = \pm i$ (05)

So complementary function is

$\boldsymbol{Y_c = (C_1 \cos x + C_2 \sin x)}$ (05)

Where C_1, C_2 are arbitrary constants.

Particular Integral is

$Y_p = \frac{1}{(D^2 + 1)}(Cos\ x + e^x \sin x + xe^{2x}$

$Y_p = \ \boldsymbol{\frac{1}{(D^2 + 1)} \cos x + \ \frac{1}{(D^2 + 1)} e^x \sin x + \frac{1}{(D^2 + 1)} xe^{2x}}$(30)

$Y = Y_p + Y_c$ (05)

(02) (a)

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 34e^{3x}$

Take the trial function

$y_T = \propto e^{3x}$

$$\frac{dy_T}{dx} = 3 \propto e^{3x}$$

$9 \propto e^{3x} + 12 \propto e^{3x} + 13 \propto e^{3x} = 34 \propto e^{3x}$

$(34 \propto - 34)e^{3x} = 0$

Since $e^{3x} \neq 0, \propto = 1, y_T = e^{3x}$ (10)

The auxiliary equation is

$(m^2 + 4m + 13) = 0$

$\therefore m = -2 \pm 3i$ (05)

The complementary function is

$Y_c = e^{-2x}(A \cos3 x + B \sin3 x)$ (05)

Where A and B are arbitrary constants.

The General solution is

$\boldsymbol{Y = e^{-2x}(Acos3x + B \sin3 x) + e^{3x}}$ (05)

(b)

Particular Integral is

$$Y_p = \frac{1}{(D^2+2D-2)} 5\sin 3x$$

$$Y_p = \frac{1}{(D+2)(D-1)} 5 sin3x$$

$$Y_p = \frac{1}{3}\left[\frac{1}{(D-1)} - \frac{1}{(D+2)}\right] 5\ sin3x$$

$$Y_p = \frac{5}{3}\left[\frac{1}{(D-1)}\sin 3x - \frac{1}{(D+2)}\ sin3x\right]$$

$$Y_p = \frac{5}{3}\left[e^x \frac{1}{D} e^{-x}\sin 3x\, dx - e^{-2x}\frac{1}{D} e^{2x}\sin 3x\, dx\right]$$

$$Y_p = \frac{5}{3}\left[e^x \int e^{-x}\sin 3x\, dx - e^{-2x}\int e^{2x}\sin 3x\, dx\right]$$

$$Y_p = \frac{5}{3}\left[e^x \frac{e^{-x}}{10}(-\sin 3x - 3cos3x) - e^{-2x}\frac{3e^{2x}}{13}(\frac{2}{3}\sin 3x - cos3x)\right]$$

$$Y_p = \frac{5}{3}\left[\frac{1}{10}(-\sin 3x - 3cos3x) - \frac{1}{13}(2\sin 3x - 3cos3x)\right]$$

$$Y_p = \frac{5}{3\times 130}[-13\sin 3x - 39cos3x - 20\sin 3x + 30cos3x]$$

$$= \frac{1}{78}(-33\sin 3x - 9cos3x)$$

$$= \frac{(-11\sin 3x - 3cos3x)}{26} \qquad (20)$$

The auxiliary equation is

$(m^2 + m - 2) = 0$

$(m+2)(m-1) = 0$

$\therefore m = -2\ and\ m = 1$ (05)

The complementary function is

$Y_c = C_1 e^{-2x} + C_2 e^x$ (05)

Where C_1 and C_2 are arbitrary constants.

The General solution is

$$\boldsymbol{Y = C_1 e^{-2x} + C_2 e^x + \frac{1}{26}(-11\sin 3x - 3cos3x)} \ (05)$$

(03)

(a)

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$= \int_0^5 e^{-st}(3)dt + \int_5^\infty e^{-st}t^2 dt$$

$$= 3\frac{e^{-st}}{-s}\Bigg|_0^5 + \int_5^\infty t^2 \frac{d}{dt}\left(\frac{-e^{-st}}{s}\right)dt$$

$$= \frac{-3}{5}(e^{-5s} - 1) + e^{-st}\frac{-t^2}{s}\Big|_5^\infty + \frac{2}{s}\int_5^\infty te^{-st}dt$$

$$= \frac{3}{5}(1 - e^{-5s}) + \frac{25}{s}(e^{-5s}) + \frac{2}{s}\left(\frac{te^{-st}}{-s}\right)_5^\infty + \frac{1}{s}\int_5^\infty e^{-st}dt$$

$$= \frac{3}{5} + \frac{22}{s}(e^{-5s}) + \frac{2}{s^2}(5e^{-5s}) - \frac{2}{s^2}\frac{e^{-st}}{-s}\Big|_5^\infty$$

$$= \frac{3}{5} + \frac{22}{s}(e^{-5s}) + \frac{10}{s^2}(e^{-5s}) + \frac{2}{s^3}(0 - e^{-5s})$$

$$= \frac{3}{5} + e^{-5s}\left(\frac{22}{s} + \frac{10}{s^2} + \frac{2}{s^3}\right) \quad (10)$$

(b)

$$L\{G(t)\} = \int_0^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}\,0\,dt + \int_a^\infty e^{-st}G(t-a)dt$$

$$Let\ u = t - a \Longrightarrow du = dt$$

$$L\{G(t)\} = \int_0^\infty e^{-s(u+a)}F(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su}F(u)dt$$

$$= e^{-as}f(s) \quad (15)$$

(c) i). $F(t) = cos^2 at$

$$L\{F(t)\} = \int_0^\infty e^{-st}cos^2 at\,dt$$

$$= \int_0^\infty e^{-st}\frac{\cos 2at + 1}{2}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st}\cos 2at\,dt + \int_0^\infty e^{-st}dt\right]$$

$$= \frac{1}{2}\left[\frac{s}{(s^2 + 4a^2)} + \frac{1}{s}\right]$$

$$= \frac{1}{2}\left(\frac{2s^2 + 4a^2}{s(s^2 + 4a^2)}\right)$$

$$= \frac{s^2 + 2a^2}{s(s^2 + 4a^2)} (15)$$

ii).F(t) = $e^{-3t} \sin 5t$

$$L\{F(t)\} = \int_0^{\infty} e^{-st} e^{-3t} sin5t\, dt$$

$$L\{sin5t\} = \frac{5}{s^2 + 25}$$

$$L\{e^{-3t} sin5t\} = \frac{5}{(s+3)^2 + 25} (15)$$

iii). F(t) = $te^{-3t} \sin 5t$

$$L\{F(t)\} = \int_0^{\infty} e^{-st} te^{-3t} sin5t\, dt$$

$$L\{e^{-3t} sin5t\} = \frac{5}{(s+3)^2 + 25}$$

We know that $f(s) = L\{F(t)\}$

$\Rightarrow f'(s) = -L\{tF(t)\}$

$$\therefore L\{te^{-3t} sin5t\} = -\frac{d}{ds}\left(\frac{5}{(s+3)^2 + 25}\right)$$

$$= -5(-1)[(s+3)^2 + 25]^{-2}.\, 2(s+3)$$

$$= \frac{10(s+3)}{[(s+3)^2 + 25]^2} (15)$$

(04)

(a) i).$y'' - 3y' - 10y = 2 \qquad , y(0) = 1, y'(0) = 2$

Take the Laplace transform on both sides:

Let L {y(t)} = y(s) , and then

$$s^2 y(s) - sy(0) - y'(0) - 3(sy(s) - 1) - 10y(s) = \frac{2}{5}$$

$$s^2 y(s) - s - 2 - 3(sy(s) - 1) - 10y(s) = \frac{2}{5}$$

$$(s^2 - 3s - 10)y(s) = \frac{2}{s} + s - 1 = \frac{s^2 - s + 2}{s}$$

$$y(s) = \frac{s^2 - s + 2}{s(s^2 - 3s - 10)}$$

$$y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}$$

$$\frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{(s-5)} + \frac{C}{(S+2)}$$

$$= \frac{A(s^2 - 3s - 10) + B(s^2 + 25) + C(s^2 - 5s)}{s(s-5)(s+2)}$$

$Set\ s = 0, We\ get\ -10A = 2, \Longrightarrow A = -1/5$

$Set\ s = 5, \qquad 35B = 22, \Rightarrow B = 22/35$

$Set\ s=-2, We\ get\ 14C=4, \Rightarrow C=2/7$

$$L'\{y(s)\}=L'\left(\frac{-1/5}{s}\right)+\ L'\left(\frac{22/35}{s-5}\right)+L'\left(\frac{2/7}{s+2}\right)$$

$$=\frac{-1}{5}+\frac{22}{35}e^{5t}+\frac{2}{7}e^{-2t}(25)$$

ii).$y''+9y'=cos2t \quad ,y(0)=C_1,y'(0)=C_2$

Take the Laplace transform on both sides:

$$s^2y(s)-sy(0)-y'(0)+9y(s)=\frac{s}{s^2+4}$$

Since $y(0)=C_1,y'(0)=C_2$

$$(s^2+9)y(s)-sC_1-C_2=\frac{s}{s^2+4}$$

$$(s^2+9)y(s)=\frac{s}{s^2+4}+sC_1+C_2=\frac{s+(5C_1+C_2)(s^2+4)}{s^2+4}$$

$$y(s)=\frac{s+(sC_1+C_2)(s^2+4)}{(s^2+4)(s^2+9)}$$

$$y(s)=\frac{1s}{5(s^2+4)}+\frac{\left(C_1-\frac{1}{5}\right)s+C_2}{(s^2+9)}$$

Taking inverse Laplace transforms,

$$y(t)=\frac{1}{5}\cos 2t+\left(C_1-\frac{1}{5}\right)\cos 3t+\frac{C_2}{3}\sin 3t\ (25)$$

(b)

$$L\{t^n\}=\frac{n!}{s^{n+1}}$$

Ifn=1,

$$L\{t\}=\int_0^\infty te^{-st}dt=\frac{te^{-st}}{-s}\Bigg|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt$$

$$=\frac{-1}{s^2}e^{-st}\Bigg|_0^\infty=\frac{1}{s^2}=R.H.S.$$

Hence true for n=1.

We assume true for n=p.

$$\therefore L\{t^p\}=\frac{p!}{s^{p+1}}$$

Then we want to show that true for n=p+1

From Laplace transform of integral (Theorem 1.5)

If $L\{F(t)\}=f(s)$, then

$$L\left\{\int_0^\infty F(t)dt\right\}=\frac{f(s)}{s}$$

Therefore,

$$L\{t^{p+1}\}=(p+1)L\{\int_0^\infty t^p dt\}$$

$$= \frac{(p+1)f(s)}{s}$$
$$= \frac{(p+1)}{s} L\{t^p\} = \frac{(p+1)}{s}\frac{p!}{S^{(p+1)}}$$
$$= \frac{(p+1)!}{s^{(p+2)}}$$
Here true for n=p+1 (20)
Hence by the mathematical induction result true for any positive integer n.

(b) i).
$$L^{-1}\left\{\frac{2s-5}{s^2-4s+15}\right\} = L^{-1}\left\{\frac{2s-5}{(s-2)^2+9}\right\}$$
$$= L^{-1}\left\{\frac{2(s-2)-1}{(s-2)^2+9}\right\}$$
$$= 2L^{-1}\left\{\frac{(s-2)}{(s-2)^2+9}\right\} - \frac{1}{3}L^{-1}\left\{\frac{3}{(s-2)^2+9}\right\}$$
$$= 2e^{2x}\cos 3x - \frac{1}{3}e^{2x}\sin 3x$$
$$= \frac{e^{2x}}{3}[6\cos 3x - \sin 3x](25)$$
ii).
$$L^{-1}\left\{\frac{1}{s(s+2)^3}\right\}$$
$$\frac{1}{s(s+2)^3} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(S+2)^2} + \frac{D}{(S+2)^3}$$
$$= A(s+2)^3 + B(s+2)^2 s + C(s+2)s + Ds$$

$Set\ s = 0, \Longrightarrow 8A = 1, \Longrightarrow A = 1/8$
$Set\ s = -2, \Longrightarrow -2D = 1, \Rightarrow D = -1/2$
$Set\ s = 1, \Longrightarrow 27A + 9B + 3C + D = 1$
$$9B + 3C = 1 + \frac{1}{2} - \frac{27}{8} = \frac{8+4-27}{8} = \frac{-15}{8}$$
$$3B + C = -\frac{5}{8} \longrightarrow (1)$$
$s = -1$
$A - B - C - D = 1$
$$-B - C = 1 - \frac{1}{2} - \frac{1}{8} = \frac{8-4-1}{8} = \frac{3}{8} \longrightarrow (2)$$
$(1) + (2) \Rightarrow$
$$2B = -\frac{2}{8} \Longrightarrow B = -\frac{1}{8}$$
$(1) \Rightarrow$
$$C = -\frac{5}{8} + \frac{3}{8} = -\frac{2}{8}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s+2)^3}\right\}$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{8}L^{-1}\left(\frac{-\frac{2}{8}}{(s+2)^2}\right)L^{-1}\left(\frac{-1/2}{(s+2)^3}\right)$$

$$= \frac{1}{8} - \frac{1}{8}e^{-2t} - \frac{2}{8}te^{-2t} - \frac{1}{2}t^2e^{-2t} \qquad (25)$$